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Note

Lower Bounds for Small Diagonal Ramsey Numbers

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Let $p = 4r + 1$ be a prime. Let G be the graph on the p points $0, 1, \ldots, p-1$ formed by connecting two points with an edge iff their difference is a quadratic residue mod p. Let k be the size of the largest clique contained in G. Then it is well known that the diagonal Ramsey number $R_2(k+1) > p$. We show $R_2(k+2) > 2p+2$. We also compute k for all $p < 3000$. © 1986 Academic Press, Inc.

The diagonal Ramsey number $R_2(k) = R(k, k)$ is the smallest integer n such that if the edges of K_n are two-colored, K_n must contain a monochromatic clique containing k vertices.

Let $p \equiv 1 \bmod 4$ be a prime. Let G_p be the graph on the p points $0, 1, \ldots, p-1$ formed by connecting two points x, y with an edge iff $x - y$ is a quadratic residue mod p. Note since $p \equiv 1 \bmod 4$, -1 is a quadratic residue mod p and $x - y$ is a quadratic residue iff $y - x$ is. Let $k = c(G_p)$ be the size of the largest clique contained in G. Since G_p is isomorphic to $\bar{G}_p$, we have $R_2(k+1) > p$. Let H_p be the graph on the $2p+2$ points $0, 1, \ldots, p-1, \lambda, 0', 1', \ldots, (p-1)', \lambda'$ containing the edges

$$\begin{array}{ll} (x, \lambda), (x', \lambda'), (x, x'), (\lambda, \lambda') & x \in \{0, 1, \ldots, p-1\} \\ (x, y), (x', y') & (x, y) \in G_p \\ (x, y'), (x', y) & (x, y) \in \bar{G}_p. \end{array}$$

Then we have

LEMMA 1. $c(H_p) = c(G_p) + 1 = k + 1$.

Proof. Let S be a clique contained in H_p. If $\{x, x'\} \subseteq S$ then $|S| = 2$ since the points x, x' have no common neigbors ($x \in \{0, 1, \ldots, p-1, \lambda\}$). Suppose $\lambda \in S$. Then since the neighbors of λ (besides λ') form a graph

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isomorphic to G_p, we have $|S| \leqslant k+1$. Similarly for $\lambda' \in S$ we have $|S| \leqslant k+1$. Hence we may assume $S = \{x_1, ..., x_m, y'_1, ..., y'_n\}$, where the x's and y's are disjoint subsets of $\{0, 1, ..., p-1\}$. Then $\{(x_1 - a), (x_2 - a), ..., (x_m - a), (y_1 - a)', (y_2 - a)', ..., (y_n - a)'\}$ is also a clique (where the subtraction is mod p). Hence we may assume $x_1 = 0$. By the definition of H_p this means $x_2, ..., x_m$ are quadratic residues mod p and $y_1, ..., y_n$ are quadratic non-residues mod p. Furthermore, the difference between any pair of x's or y's must be a quadratic residue while the difference between an x and a y must be a quadratic non-residue. Let $T = \{1/x_2, ..., 1/x_m, 1/y_1, ..., 1/y_n\}$ (again arithmetic is mod p). Then it is easy to verify that T is a clique in G_p. For example, $1/x_i - 1/y_j = (y_j - x_i)/(x_i y_j)$, which is a quadratic residue since x_i is a quadratic residue while $(y_j - x_i)$ and y_j are quadratic non-residues. The other cases are similar. Hence, $|S| = |T| + 1 \leqslant k+1$. Therefore, $c(H_p) \leqslant k+1$. However, by adjoining λ to a maximum clique in G_p we have $c(H_p) \geqslant k+1$. Therefore $c(H_p) = k+1$.

THEOREM 1. *If* $c(G_p) = p$ *then* $R_2(k+2) > 2p+2$.

Proof. $\bar{H}_p$ is isomorphic to H_p with the edges $(x, x')\, x \in \{0, 1, ..., p-1, \lambda\}$ deleted. Hence $c(\bar{H}_p) \leqslant c(H_p)$. The theorem follows at once from the lemma.

Theorem 1 allows us to improve the lower bounds for $R_2(k)$ for some small values of k. For example, $c(G_{101}) = 5$. Hence $R_2(7) > 204$. The best

TABLE I

k	p_1	p_2	n
2	5	5	1
3	13	17	2
4	29	37	2
5	41	101	6
6	97	109	2
7	113	281	10
8	173	373	7
9	229	797	15
10	557	709	3
11	433	1277	32
12	613	1493	13
13	853	2741	53
14	1373	2801	17
15	1289		38
16	2389		4
17	1741		5
18			0
19	2729		1

previous lower bound known to the author is $R_2(7) > 125$ [2]. There remains a considerable gap between the lower bound and the upper bound $R_2(7) \leqslant 586$ [2].

Unfortunately, the behavior of $c(G_p)$ for large p is unknown and appears to be a difficult problem. We have computed $c(G_p)$ for all primes $p \equiv 1 \bmod 4$, $p \leqslant 3000$. Our results are summarized in Table I. In Table I, for each k, p_1, p_2, and n are the smallest, largest, and number of such primes p we found with $c(G_p) = k$. We have also shown $c(G_p) \geqslant 15$ for $3000 < p < 10000$, $p \equiv 1 \bmod 4$. There seems to be a definite tendency for $c(G_p)$ to be odd which the author is unable to explain.

Our program is written in Fortran and requires several minutes of CPU time on an IBM 3081 K to find $c(G_p)$ for p near 3000.

Finally, we remark that a similar result holds for qth residue graphs which we state without proof. Let p be a prime such that $p \equiv 1 \bmod 2q$. Let G_p be the graph on the p points $0, 1, ..., p-1$ formed by connecting two points with an edge iff their difference is a qth residue mod p. Let $R_q(k)$ be the smallest integer n such that if the edges of K_n are q colored, K_n must contain a monochromatic clique containing k vertices. Let $k = c(G_p)$. Then we have $R_q(k+1) > p$, $R_q(k+2) > q(p+1)$.

References

1. J. P. Burling and S. W. Reyner, Some lower bounds of the Ramsey numbers $n(k, k)$, *J. Combin. Theory Ser. B* **13** (1972), 168–169.
2. F. R. K. Chung and C. M. Grinstead, A survey of bounds for classical Ramsey numbers, *J. Graph Theory* **7** (1983), 25–37.